

THE GENERALIZED REMAINDER AND QUOTIENT THEOREMS OF UNIVARIATE POLYNOMIALS

WIWAT WANICHARPICHAT

ABSTRACT. The author in [7] was proved the generalized remainder and quotient theorems of polynomial in one indeterminate where the divisor is complete factorization to linear factors. In this paper we give the formula for the generalized remainder theorem and the generalized quotient theorem of polynomials when the divisor is not factorization to linear factors.

1. INTRODUCTION AND PRELIMINARY

Let $f(x)$ and $g(x)$ be polynomials over a field F . Since $F[x]$ is a Euclidean domain, the Division Algorithm holds and hence there are (unique) polynomials $q(x)$ and $r(x)$ in $F[x]$ such that

$$f(x) = g(x)q(x) + r(x)$$

where $r(x) = 0$ or its degree is less than that of $g(x)$. If $g(x) = x - b$, the well known Remainder Theorem gives $r(x) = f(b)$. The author in [6] was shown that, if $g(x) = ax - b$, $a \neq 0$, then a simple extension to this result gives the remainder as $f(a^{-1}b)$. This is true since, $f(x) = (x - a^{-1}b)q(x) + f(a^{-1}b)$, by the remainder theorem, for some polynomial $q(x)$ and by taking out a constant a and writing $q(x) = aq_1(x)$, we have $f(x) = (ax - b)q_1(x) + f(a^{-1}b)$. Thus, the remainder term does not change and it is sufficient to consider divisors which are monic polynomials. This result may be extended to divisors which are polynomials of degree higher than 1. If $f(x)$ is a polynomial of degree n and $g(x)$ is a polynomial of degree m , the remainder polynomial $r(x)$ is unchanged if $g(x)$ is replaced by the corresponding monic polynomial obtained from it by taking out the leading coefficient.

Lemma 1.1 ([7, Lemma 2.1]). *Let $f(x)$ and $g(x) = b_mx^m + b_{m-1}x^{m-1} + \cdots + b_1x + b_0$ be polynomials in $F[x]$, $\deg g(x) = m$ and $b_m \neq 1$, if*

$$g_1(x) = \frac{1}{b_m}g(x) = x^m + \beta_{m-1}x^{m-1} + \cdots + \beta_1x + \beta_0$$

where $\beta_i = \frac{b_i}{b_m}$, $i = 0, 1, \dots, m-1$, be the corresponding monic polynomial of $g(x)$, and $q_1(x)$, $q(x)$ be the quotients and $r_1(x)$, $r(x)$ be the remainders on dividing $f(x)$ by $g_1(x)$, and by $g(x)$ respectively, then

$$(1.1) \quad q(x) = \frac{1}{b_m}q_1(x) \quad \text{and} \quad r(x) = r_1(x).$$

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Proof. Since

$$g(x) = b_0 + b_1x + \dots + b_{m-1}x^{m-1} + b_mx^m, \quad b_m \neq 1.$$

Let

$$\begin{aligned} g_1(x) &= \frac{1}{b_m}g(x) \\ &= \frac{b_0}{b_m} + \frac{b_1}{b_m}x + \dots + \frac{b_{m-1}}{b_m}x^{m-1} + x^m. \end{aligned}$$

By the Division Algorithm, there is a unique $q_1(x)$, and a unique $r_1(x)$ in $F[x]$ such that

$$f(x) = q_1(x)g_1(x) + r_1(x) \text{ whenever } r_1(x) = 0 \text{ or } \deg r_1(x) < \deg g_1(x),$$

that is

$$\begin{aligned} f(x) &= q_1(x)(\beta_0 + \beta_1x + \dots + \beta_{m-1}x^{m-1} + x^m) + r_1(x) \\ &= q_1(x)\left(\frac{b_0}{b_m} + \frac{b_1}{b_m}x + \dots + \frac{b_{m-1}}{b_m}x^{m-1} + x^m\right) + r_1(x) \\ &= q_1(x)\frac{1}{b_m}(b_0 + b_1x + \dots + b_{m-1}x^{m-1} + b_mx^m) + r_1(x) \\ &= \left\{\frac{1}{b_m}q_1(x)\right\}g(x) + r_1(x) \\ &= q(x)g(x) + r(x). \end{aligned}$$

Thus

$$q(x) = \frac{1}{b_m}q_1(x), \text{ and } r(x) = r_1(x).$$

□

Although there are many computational algorithms available for obtaining the coefficients of the quotient and remainder polynomials, there is to-date, no explicit algebraic formula that can be used to compute them. Perhaps, this is because existing algorithms are so computationally efficient. However, computers are incredibly fast nowadays, so that no one is going to complain if an algorithm gives the result in seconds rather than in milliseconds. On the other hand, knowing that a problem can be handled in finite time does not mean we know how long it will take, and so, there will always be a need for efficient algorithms. An interesting discussion of this point, among other things, can be found in [5].

In the present paper, the existence of an algebraic formula will allow the extension of the use of the Division Algorithm into areas that have hitherto not been contemplated. To illustrate this, we consider a novel application where the algebraic formula for the coefficients of the general remainder and quotient polynomial are used to compute a lower Hessenberg-Toeplitz matrices.

2. THE GENERALIZED QUOTIENT THEOREM

If $f(x) = a_nx^n + a_{n-1}x^{n-1} + \dots + a_2x^2 + a_1x + a_0$ where $a_n \neq 0$, and $g(x) = x^m + b_{m-1}x^{m-1} + \dots + b_2x^2 + b_1x + b_0$, is a monic polynomial with $m < n$ then there exist polynomials $q(x) = d_{n-m}x^{n-m} + d_{n-m-1}x^{n-m-1} + \dots + d_2x^2 + d_1x + d_0$ and $r(x) = c_{m-1}x^{m-1} + \dots + c_1x + c_0$ such that

$$(2.1) \quad f(x) = g(x)q(x) + r(x).$$

Equating the coefficients in (2.1) leads to the following two systems of linear equations. The first system of m linear equations is given by

$$(2.2) \quad c_k + \sum_{i+j=k} b_id_j = a_k$$

where $k = 0, 1, \dots, m-1$; $i = 0, 1, \dots, m$; and $j = 0, 1, \dots, n-m$. The second system of $n-m$ linear equations is of the form:

$$(2.3) \quad a_k = \sum_{i+j=k} b_i d_j$$

This holds for $k = m, m+1, \dots, n$, the ranges for i and j remaining unchanged. Writing (2.3) out, we get

$$(2.4) \quad \begin{array}{rcl} b_{2m-n}d_{n-m} + b_{2m-n+1}d_{n-m-1} + \dots + b_{m-1}d_1 + d_0 & = & a_m, \\ b_{2m-n+1}d_{n-m} + b_{2m-n+2}d_{n-m-1} + \dots + d_1 & = & a_{m+1}, \\ & \vdots & \\ b_{m-1}d_{n-m} + d_{n-m-1} & = & a_{n-1}, \\ d_{n-m} & = & a_n, \end{array}$$

with the proviso that $b_m = 1$ and $b_i = 0$ when i is negative, b_i will have negative subscript when $m < \frac{n}{2}$. The system of linear equations (2.4) allows us to compute $d_{n-m}, d_{n-m-1}, \dots, d_0$ recursively, giving,

$$(2.5) \quad \begin{array}{rcl} d_{n-m} & = & a_n, \\ d_{n-m-1} & = & a_{n-1} - b_{m-1}a_n, \\ & \vdots & \\ d_1 & = & a_{m+1} - b_{2m-n+1}a_n - b_{2m-n+2}(a_{n-1} - b_{m-1}a_n) - \dots - b_{m-1}d_2, \\ d_0 & = & a_m - b_{2m-n}a_n + b_{2m-n+1}(a_{n-1} + b_{m-1}a_n) + \dots \\ & & + b_{m-1}(a_{n-1} - b_{2m-n+1}a_n - b_{2m-n+2}(a_{n-1} - b_{m-1}a_n) - \dots - b_{m-1}d_2). \end{array}$$

The coefficients d_i , $i = 0, 1, \dots, n-m$, can be written in terms of the elements of the linear recurrent sequence $\{s_n\}$, where

$$s_{n+r} = \alpha_0 s_n + \alpha_1 s_{n+1} + \dots + \alpha_{r-1} s_{n+r-1},$$

for positive integers n and r , and $\alpha_i \in F$, $i = 0, 1, \dots, r-1$. The result is summarized in Theorem 2.1 below.

Theorem 2.1. *If $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0$ where $a_n \neq 0$, and $g(x) = x^m + b_{m-1} x^{m-1} + \dots + b_2 x^2 + b_1 x + b_0$, are polynomials in $F[x]$ with $m < n$, then, the quotient, on dividing $f(x)$ by $g(x)$ is $q(x) = d_{n-m} x^{n-m} + d_{n-m-1} x^{n-m-1} + \dots + d_2 x^2 + d_1 x + d_0$ where*

$$(2.6) \quad d_{n-m-k} = \sum_{i=0}^k s_{k+1-i} a_{n-i}$$

and $\{s_n\}$ is the linear recurrent sequence, $s_1 = 1$ and $s_r = \sum_{i=1}^{r-1} b_{m-i} s_{r-i}$ for $r = 2, 3, \dots$

Proof. The theorem can be proved by mathematical induction on k . It is obviously true for $k = 0$.

We assume that it is true for all $d_{n-m-\nu}$, $\nu \leq k < n-m$, so that

$$(2.7) \quad d_{n-m-\nu} = \sum_{i=0}^{\nu} s_{\nu+1-i} a_{n-i}, \quad \text{for all } \nu \leq k.$$

From (2.4), we have

$$a_{n-\nu-1} = -b_m d_{n-m-\nu-1} + b_{m-1} d_{n-m-\nu} + b_{m-2} d_{n-m-\nu+1} + \dots + b_{m-\nu+1} d_{n-m},$$

$b_m = 1$, so that

$$d_{n-m-\nu-1} = a_{n-\nu-1} - b_{m-1}d_{n-m-\nu} - b_{m-2}d_{n-m-\nu+1} + \dots - b_{m-\nu-1}d_{n-m}.$$

Substituting from (2.7) we get

$$d_{n-m-\nu-1} = a_{n-\nu-1} - b_{m-1} \sum_{i=0}^{\nu} s_{\nu+1-i} a_{n-i} - b_{m-2} \sum_{i=0}^{\nu-1} s_{\nu-i} a_{n-i} - \dots - b_{m-\nu-1} s_1 a_n.$$

Expanding the summation terms and regrouping we have

$$d_{n-m-\nu-1} = \sum_{i=0}^{\nu+1} s_{\nu+2-i} a_{n-i}.$$

Hence (2.6) is true for all $k \leq n - m$. \square

By Lemma 1.1, we can now extend the above result to the case where b_m may not equal 1.

Corollary 2.2 (Generalized Quotient Theorem). *If $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$, $g_1(x) = b_m x^m + b_{m-1} x^{m-1} + \dots + b_1 x + b_0$ are polynomials in $F[x]$ of degree n and m respectively where $n > m$, the quotient when $f(x)$ is divided by $g_1(x)$ is given by*

$$q_1(x) = \bar{d}_{n-m} x^{n-m} + \bar{d}_{n-m-1} x^{n-m-1} + \dots + \bar{d}_1 x + \bar{d}_0$$

where $\bar{d}_{n-m-k} = \frac{1}{b_m} \sum_{j=0}^k t_{k+1-j} a_{n-j}$ and $t_1 = \frac{1}{b_m}$, $t_r = \frac{1}{b_m} \sum_{i=1}^{r-1} b_{m-i} t_{r-i}$ for $r = 2, 3, \dots$

Proof. By letting $g(x) = x^m + b'_{m-1} x^{m-1} + b'_{m-2} x^{m-2} + \dots + b'_1 x + b'_0$ where $b'_{m-i} = b_{m-i}/b_m$, the quotient, when $f(x)$ is divided by $g(x)$, is given by

$$q(x) = d'_{n-m} x^{n-m} + d'_{n-m-1} x^{n-m-1} + \dots + d'_1 x + d'_0$$

where we note that the sequence $\{t_k\}$ is identical to $\{s_k\}$ of Theorem 2.1 when $b_m = 1$ and hence $d'_{n-m-k} = \sum_{i=0}^k t_{k+1-i} a_{n-i}$, $t_1 = \frac{1}{b_m}$, and $t_r = \sum_{i=1}^{r-1} b'_{m-i} t_{r-i} = \frac{1}{b_m} \sum_{i=1}^{r-1} b_{m-i} t_{r-i}$. Now $g(x) = g_1(x)/b_m$ so that the quotient $q_1(x)$, when $f(x)$ is divided by $g_1(x)$ is $q(x)/b_m$ where $\bar{d}_{n-m-k} = \frac{1}{b_m} \sum_{j=0}^k t_{k+1-j} a_{n-j}$. \square

Since the coefficients a_0, a_1, \dots, a_{m-1} of $f(x)$ do not contribute to the quotient polynomial, the following corollary may be stated without proof.

Corollary 2.3. *If $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$, $g(x) = b_m x^m + b_{m-1} x^{m-1} + \dots + b_1 x + b_0$ are polynomials in $F[x]$ of degree n and m respectively where $n > m$, suppose $q(x)$ is the quotient on dividing $f(x)$ by $g(x)$, and $\hat{q}(x)$ is the quotient on dividing $\hat{f}(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_m x^m$ by $g(x)$ then $q(x) = \hat{q}(x)$.*

3. THE GENERALIZED REMAINDER THEOREM

The remainder $r(x)$ can now be obtained from (2.2). Since the coefficients c_k depends on the coefficients of the quotient polynomial, the most general expression for the remainder can be obtained when the divisor is a general polynomial of degree $m < n$. The result is given by the next theorem.

Theorem 3.1 (Generalized Remainder Theorem). *If $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0$, $g(x) = b_m x^m - b_{m-1} x^{m-1} - \dots - b_2 x^2 - b_1 x - b_0$ where $a_n \neq 0$ and $b_m \neq 0$, are polynomials in $F[x]$ and suppose that $m \leq n$, then the remainder on dividing $f(x)$ by $g(x)$ is*

$$(3.1) \quad r(x) = \sum_{k=0}^{m-1} \left(a_k + \frac{1}{b_m} \sum_{i+j=k} b_i \sum_{\nu=0}^{n-m-j} t_{n-m-j+1-\nu} a_{n-\nu} \right) x^k$$

where $\{t_r\}$ is the linear recurrent sequence, $t_1 = \frac{1}{b_m}$ and $t_r = \frac{1}{b_m} \sum_{i=1}^{r-1} b_{m-i} t_{r-i}$ for $r = 2, 3, \dots$.

Proof. From the system of linear equations (2.2),

$$c_k = a_k + \sum_{i+j=k} b_i d_j$$

where $k = 0, 1, \dots, m-1$; $i = 0, 1, \dots, m$; and $j = 0, 1, \dots, n-m$.

Using the result of Corollary 2.2, and putting $n-m-k = j$ there, we get

$$d_j = \frac{1}{b_m} \sum_{\nu=0}^{n-m-j} t_{n-m-j+1-\nu} a_{n-\nu}.$$

Hence

$$r(x) = \sum_{k=0}^{m-1} \left(a_k + \frac{1}{b_m} \sum_{i+j=k} b_i \sum_{\nu=0}^{n-m-j} t_{n-m-j+1-\nu} a_{n-\nu} \right) x^k.$$

□

We note, in passing, that $t_{n-m-j+1-\nu}$ always has positive suffices, by construction.

4. A LOWER HESSENBERG-TOEPLITZ MATRIX

Toeplitz matrices is a class of matrices, whose elements along each diagonal are the same constants, see [4, p.27] and [2, p.1]. Elouafi and Aiat Hadj [3, pp.177-178], defined the lower Hessenberg matrix as follows: The matrix $\mathbf{H} = (h_{ij})_{1 \leq i, j \leq n}$ is called lower Hessenberg matrix if $h_{ij} = 0$ for $i < j+1$, and assume that all elements of the super diagonal of the lower Hessenberg matrix to be non-zero, they given a recursion formula for the characteristic polynomial of the matrix \mathbf{H} . Now we defined the lower Hessenberg-Toeplitz matrices of order k , which have the form

$$\Delta_k = \begin{vmatrix} -b_{m-1} & b_m & 0 & \dots & 0 \\ -b_{m-2} & -b_{m-1} & b_m & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ -b_{m-k+1} & -b_{m-k+2} & & \ddots & b_m \\ -b_{m-k} & -b_{m-k+1} & -b_{m-k+2} & \dots & -b_{m-1} \end{vmatrix}$$

where $b_m \neq 0$. In this paper we attempt to find the generalized remainder and quotient theorems which related to the determinants of Hessenberg-Toeplitz matrices of this form.

By this second system of linear equations (2.4) where $b_m \neq 0$, we get a matrix equation $\mathbf{H}\mathbf{d} = \mathbf{a}$,

$$(4.1) \quad \begin{pmatrix} b_{2m-n} & b_{2m-n+1} & \cdots & b_{m-1} & b_m \\ b_{2m-n+1} & & \ddots & b_m & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ b_{m-1} & b_m & \ddots & & \vdots \\ b_m & 0 & \cdots & \cdots & 0 \end{pmatrix} \begin{pmatrix} d_{n-m} \\ d_{n-m-1} \\ \vdots \\ d_1 \\ d_0 \end{pmatrix} = \begin{pmatrix} a_m \\ a_{m+1} \\ \vdots \\ a_{n+1} \\ a_n \end{pmatrix}.$$

The square backward upper triangular matrix \mathbf{H} of order $k = n - m + 1$ is a Hankel matrix with all entries on the backward diagonal as the leading coefficient b_m of $g(x)$. So for some $b_j = 0$ for $m < \frac{n}{2}$. If then $m < \frac{n}{2}$ for some subscript j of the entries b_j in \mathbf{H} is a negative we define $b_j = 0$ if $j < 0$, in this case the matrices equation (4.1) is of the form:

$$(4.2) \quad \begin{pmatrix} 0 & \cdots & 0 & b_0 & \cdots & b_{m-1} & b_m \\ \vdots & \ddots & \ddots & & \ddots & \ddots & 0 \\ 0 & \ddots & & \ddots & \ddots & \ddots & \vdots \\ b_0 & & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & & \vdots \\ b_{m-1} & b_m & \ddots & & & & \vdots \\ b_m & 0 & \cdots & \cdots & \cdots & \cdots & 0 \end{pmatrix} \begin{pmatrix} d_{n-m} \\ d_{n-m-1} \\ \vdots \\ \vdots \\ \vdots \\ d_1 \\ d_0 \end{pmatrix} = \begin{pmatrix} a_m \\ a_{m+1} \\ \vdots \\ \vdots \\ \vdots \\ a_{n-1} \\ a_n \end{pmatrix}.$$

Now, let \mathbf{W} be a bordered matrix obtained from \mathbf{H} and \mathbf{a} see [1, p.417], defined as follows:

$$(4.3) \quad \mathbf{W} = \begin{pmatrix} \mathbf{H} & \mathbf{a} \\ \mathbf{x}^T & 0 \end{pmatrix}$$

where the matrix \mathbf{H} and \mathbf{a} is define in (4.1) and

$$\mathbf{x}^T = (x^{n-m} \quad x^{n-m} \quad \cdots \quad x^2 \quad x \quad 1).$$

Thus \mathbf{W} is a square matrix of order $t = k + 1 = n - m + 2$.

From (4.3)

$$(4.4) \quad \det(\mathbf{W}) = \begin{vmatrix} \mathbf{H} & \mathbf{a} \\ \mathbf{x}^T & 0 \end{vmatrix} = \begin{vmatrix} b_{2m-n} & b_{2m-n+1} & \cdots & b_{m-1} & b_m & a_m \\ b_{2m-n+1} & & \ddots & b_m & 0 & a_{m+1} \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ b_{m-1} & b_m & \ddots & & \vdots & a_{n-1} \\ b_m & 0 & \cdots & \cdots & 0 & a_n \\ x^{n-m} & x^{n-m-1} & \cdots & x & 1 & 0 \end{vmatrix}.$$

Interchange the last column with the consecutive columns of the matrix \mathbf{W} until to the first column is reached and similarly interchange the last row of the matrix \mathbf{W} with the consecutive row until to the top row is reached. The sum of the number of interchange is even. Thus the new matrix, namely \mathbf{T} form from the matrix \mathbf{W}

has order t and has the same determinant as \mathbf{W} . That is

$$(4.5) \quad \mathbf{T} = \begin{pmatrix} 0 & x^{n-m} & x^{n-m-1} & \dots & x & 1 \\ a_m & b_{2m-n} & b_{2m-n+1} & \dots & b_{m-1} & b_m \\ a_{m-1} & b_{2m-n+1} & & \ddots & b_m & 0 \\ \vdots & \vdots & & \ddots & \ddots & \vdots \\ a_{n-1} & b_{m-1} & b_m & \ddots & & \vdots \\ a_n & b_m & 0 & \dots & \dots & 0 \end{pmatrix},$$

and

$$(4.6) \quad \det(\mathbf{W}) = \begin{vmatrix} 0 & x^{n-m} & x^{n-m-1} & \dots & x & 1 \\ a_m & b_{2m-n} & b_{2m-n+1} & \dots & b_{m-1} & b_m \\ a_{m+1} & b_{2m-n+1} & & \ddots & b_m & 0 \\ \vdots & \vdots & & \ddots & \ddots & \vdots \\ a_{n+1} & b_{m-1} & b_m & \ddots & & \vdots \\ a_n & b_m & 0 & \dots & \dots & 0 \end{vmatrix} = \det(\mathbf{T}).$$

Now consider a backward identity matrix (or anti-identity matrix) of order $t = n - m + 2$, \mathbf{P}_t say,

$$\mathbf{P}_t = \mathbf{P}_{n-m+2} = \begin{pmatrix} 0 & \dots & \dots & \dots & 0 & 1 \\ \vdots & & & \ddots & 1 & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ 0 & 1 & \ddots & & & \vdots \\ 1 & 0 & \dots & \dots & \dots & 0 \end{pmatrix}_{(n-m+2, n-m+2)}.$$

The following lemma and its proof are well known. We present this material here because we refer to it a few times in the next two theorems.

Lemma 4.1. *If \mathbf{P}_t (as above) is the back word identity matrix of order t , then*

$$(4.7) \quad \det(\mathbf{P}_t) = (-1)^{\frac{t(t-1)}{2}}.$$

Now consider the product of the matrix \mathbf{T} from (4.5) and \mathbf{P}_t defined as above, [4], assert that $\mathbf{P}_t \mathbf{T} = \mathbf{L}$ (and $\mathbf{T} \mathbf{P}_t = \mathbf{U}$) where \mathbf{L} is a lower Hessenberg-Toeplitz

matrix (and \mathbf{U} is an upper Hessenberg-Toeplitz matrix). In general

$$\begin{aligned}
\mathbf{P}_t \mathbf{T} &= \begin{pmatrix} 0 & \dots & \dots & \dots & 0 & 1 \\ \vdots & & & \ddots & 1 & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ 0 & 1 & \ddots & & & \vdots \\ 1 & 0 & \dots & \dots & \dots & 0 \end{pmatrix} \\
&\times \begin{pmatrix} 0 & x^{n-m} & x^{n-m-1} & \dots & x & 1 \\ a_m & b_{2m-n} & b_{2m-n+1} & \dots & b_{m-1} & b_m \\ a_{m-1} & b_{2m-n+1} & & \ddots & b_m & 0 \\ \vdots & \vdots & & \ddots & \ddots & \vdots \\ a_{n-1} & b_{m-1} & b_m & \ddots & & \vdots \\ a_n & b_m & 0 & \dots & \dots & 0 \end{pmatrix} \\
&= \begin{pmatrix} a_n & b_m & 0 & \dots & \dots & 0 \\ a_{n-1} & b_{m-1} & b_m & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ a_{m-1} & b_{2m-n+1} & & \ddots & b_m & 0 \\ a_m & b_{2m-n} & b_{2m-n+1} & \dots & b_{m-1} & b_m \\ 0 & x^{n-m} & x^{n-m-1} & \dots & x & 1 \end{pmatrix} =: \mathbf{H}_t.
\end{aligned}$$

Let $\Delta_i := \det(\mathbf{H}_i)$. Expanding the determinant of Hessenberg-Toeplitz matrix, $\det(\mathbf{H}_t)$

$$(4.8) \quad \det(\mathbf{H}_t) = \begin{vmatrix} a_n & b_m & 0 & \dots & \dots & 0 \\ a_{n-1} & b_{m-1} & b_m & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ a_{m-1} & b_{2m-n+1} & & \ddots & b_m & 0 \\ a_m & b_{2m-n} & b_{2m-n+1} & \dots & b_{m-1} & b_m \\ 0 & x^{n-m} & x^{n-m-1} & \dots & x & 1 \end{vmatrix},$$

by the last row. Thus

$$\begin{aligned}
(4.9) \quad \det(\mathbf{H}_t) &= 1\Delta_{t-1} - xb_m\Delta_{t-2} + x^2b_m^2\Delta_{t-3} - x^3b_m^3\Delta_{t-4} + \dots - x^{n-m}b_m^{n-m}\Delta_1 + 0 \\
&= \sum_{i=2}^t (-1)^{t-i} x^{t-i} b_m^{t-i} \Delta_{i-1}
\end{aligned}$$

where

$$\Delta_1 = |a_n|, \quad \Delta_2 = \begin{vmatrix} a_n & b_m \\ a_{n-1} & b_{m-1} \end{vmatrix}, \quad \Delta_3 = \begin{vmatrix} a_n & b_m & 0 \\ a_{n-1} & b_{m-1} & b_m \\ a_{n-2} & b_{m-2} & b_{m-1} \end{vmatrix}, \dots$$

$$(4.10) \quad \Delta_{t-1} = \begin{vmatrix} a_n & b_m & 0 & \dots & 0 \\ a_{n-1} & b_{m-1} & b_m & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ a_{m+1} & b_{2m-n+1} & & \ddots & b_m \\ a_m & b_{2m-n} & b_{2m-n+1} & \dots & b_{m-1} \end{vmatrix}.$$

Theorem 4.2. *If \mathbf{T} and \mathbf{H}_t are matrices of order t defined as above then*

$$(4.11) \quad \det(\mathbf{T}) = \det(\mathbf{W}) = (-1)^{\frac{t(t-1)}{2}} \det(\mathbf{H}_t)$$

Proof. Since $\det(\mathbf{H}_t) = \det(\mathbf{P}_t \mathbf{T}) = \det(\mathbf{P}_t) \det(\mathbf{T})$ implies that $\det(\mathbf{T}) = \frac{\det(\mathbf{H}_t)}{\det(\mathbf{P}_t)}$.

From (4.6) $\det(\mathbf{T}) = \det(\mathbf{W})$ we have $\det(\mathbf{W}) = \frac{\det(\mathbf{H}_t)}{\det(\mathbf{P}_t)}$. Since, $\det(\mathbf{P}_t) = (-1)^{\frac{t(t-1)}{2}}$, by Lemma 4.1. Thus (4.11) true. \square

Corollary 4.3. $\det(\mathbf{W}) = (-1)^{\frac{t(t-1)}{2}} \sum_{i=2}^t (-1)^{t-i} x^{t-i} b_m^{t-i} \Delta_{i-1}.$

Proof. By (4.11) and (4.9). \square

Theorem 4.4. *The determinant of the Hankel matrix,*

$$\mathbf{H} = \begin{pmatrix} b_{2m-n} & b_{2m-n+1} & \dots & b_{m-1} & b_m \\ b_{2m-n+1} & & \ddots & b_m & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ b_{m-1} & b_m & & \ddots & \vdots \\ b_m & 0 & \dots & \dots & 0 \end{pmatrix}_{(t-1, t-1)}.$$

is $\det(\mathbf{H}) = (-1)^{\frac{(t-1)(t-2)}{2}} b_b^{t-1}.$

Proof. Since

$$\begin{aligned} \mathbf{H} \mathbf{P}_{t-1} &= \begin{pmatrix} b_{2m-n} & b_{2m-n+1} & \dots & b_{m-1} & b_m \\ b_{2m-n+1} & & \ddots & b_m & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ b_{m-1} & b_m & & \ddots & \vdots \\ b_m & 0 & \dots & \dots & 0 \end{pmatrix} \begin{pmatrix} 0 & \dots & \dots & 0 & 1 \\ \vdots & & \ddots & 1 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 1 & \ddots & & \vdots \\ 1 & 0 & \dots & \dots & 0 \end{pmatrix} \\ &= \begin{pmatrix} b_m & b_{m-1} & \dots & b_{2m-n+1} & b_{2m-n} \\ 0 & b_m & \ddots & & b_{2m-n+1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & b_m & b_{m-1} \\ 0 & \dots & \dots & 0 & b_m \end{pmatrix} =: \mathbf{U}. \end{aligned}$$

Since $\det(\mathbf{H} \mathbf{P}_{t-1}) = \det(\mathbf{H}) \det(\mathbf{P}_{t-1}) = \det(\mathbf{U}) = b_m^{t-1}$. Thus,

$$\det(\mathbf{H}) = \frac{\det \mathbf{U}}{\det(\mathbf{P}_{t-1})}.$$

By Lemma 4.1,

$$\det(\mathbf{P}_{t-1}) = (-1)^{\frac{(t-1)[(t-1)-1]}{2}} = (-1)^{\frac{(t-1)(t-2)}{2}}.$$

Therefore:

$$\det(\mathbf{H}) = \frac{\det \mathbf{U}}{\det(\mathbf{P}_{t-1})} = \frac{b_m^{t-1}}{(-1)^{\frac{(t-1)(t-2)}{2}}} = (-1)^{\frac{(t-1)(t-2)}{2}} b_m^{t-1}.$$

The theorem was proved. \square

Theorem 4.5. *The quotient of the polynomial $f(x)$ by $g(x)$ is $q(x) = -\frac{\det(\mathbf{W})}{\det(\mathbf{H})}$.*

Proof. From (4.4)

$$\begin{aligned} \det(\mathbf{W}) &= \begin{vmatrix} \mathbf{H} & \mathbf{a} \\ \mathbf{x}^T & 0 \end{vmatrix} \\ &= \begin{vmatrix} b_{2m-n} & b_{2m-n+1} & \cdots & b_{m-1} & b_m & a_m \\ b_{2m-n+1} & & \ddots & b_m & 0 & a_{m+1} \\ \vdots & & \ddots & \ddots & \vdots & \vdots \\ b_{m-1} & b_m & \ddots & & \vdots & a_{n-1} \\ b_m & 0 & \cdots & \cdots & 0 & a_n \\ x^{n-m} & x^{n-m-1} & \cdots & x & 1 & 0 \end{vmatrix} \end{aligned}$$

and J.W. Archbold [1, p.417], assert that:

$$(4.12) \quad \det(\mathbf{W}) = -\mathbf{x}^T \text{adj}(\mathbf{H})\mathbf{a} + 0 \det(\mathbf{H}) = -\mathbf{x}^T \text{adj}(\mathbf{H})\mathbf{a}.$$

Consider (4.1), the matrix equation $\mathbf{H}\mathbf{d} = \mathbf{a}$, we get $\mathbf{d} = \mathbf{H}^{-1}\mathbf{a}$, and Theorem 4.4, show that \mathbf{H} is nonsingular matrix, thus \mathbf{H}^{-1} exists. From the inverse formula $\mathbf{H}^{-1} = \frac{1}{\det(\mathbf{H})} \text{adj}(\mathbf{H})$ we get:

$$\mathbf{d} = \frac{1}{\det(\mathbf{H})} \text{adj}(\mathbf{H})\mathbf{a}.$$

That is:

$$\mathbf{x}^T \mathbf{d} = \mathbf{x}^T \left(\frac{1}{\det(\mathbf{H})} \text{adj}(\mathbf{H})\mathbf{a} \right) = \frac{1}{\det(\mathbf{H})} (\mathbf{x}^T \text{adj}(\mathbf{H})\mathbf{a})$$

Since $q(x) = \mathbf{x}^T \mathbf{d}$ so that:

$$(4.13) \quad q(x) = \frac{1}{\det(\mathbf{H})} (\mathbf{x}^T \text{adj}(\mathbf{H})\mathbf{a})$$

From (4.12)

$$\begin{aligned} \det(\mathbf{W}) &= -\mathbf{x}^T \text{adj}(\mathbf{H})\mathbf{a} \\ &= -\det(\mathbf{H}) \left(\frac{1}{\det(\mathbf{H})} (\mathbf{x}^T \text{adj}(\mathbf{H})\mathbf{a}) \right) \\ &= -\det(\mathbf{H}) q(x). \end{aligned}$$

Therefore $q(x) = -\frac{\det(\mathbf{W})}{\det(\mathbf{H})}$. \square

Theorem 4.6. *If $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0$ and $g(x) = b_m x^m - b_{m-1} x^{m-1} - \dots - b_2 x^2 - b_1 x - b_0$ where $a_n \neq 0$, and suppose that $m = \deg g(x) \leq \deg f(x) = n$ then the quotient on dividing $f(x)$ by $g(x)$ is*

$$q(x) = \sum_{j=0}^{n-m} \left((-1)^{n-m-j} b_m^{j-(n-m+1)} \Delta_{(n-m+1)-j} \right) x^j,$$

where $\Delta_{(n-m+1)-j}$ define as in (4.10).

Proof. From Corollary 4.3, $\det(\mathbf{W}) = (-1)^{\frac{t(t-1)}{2}} \sum_{i=2}^t (-1)^{t-i} x^{t-i} b_m^{t-i} \Delta_{i-1}$ and Theorem 4.5, $q(x) = -\frac{\det(\mathbf{W})}{\det(\mathbf{H})}$, and Theorem 4.4, $\det(\mathbf{H}) = (-1)^{\frac{(r-1)(t-2)}{2}} b_m^{t-1}$ we have:

$$\begin{aligned} q(x) &= -\frac{(-1)^{\frac{t(t-1)}{2}} \sum_{i=2}^t (-1)^{t-i} x^{t-i} b_m^{t-i} \Delta_{i-1}}{(-1)^{\frac{(r-1)(t-2)}{2}} b_m^{t-1}} \\ &= (-1)^{[1+\frac{t(t-1)}{2}-\frac{(r-1)(t-2)}{2}]} b_m^{-(t-1)} \sum_{i=2}^t (-1)^{t-i} x^{t-i} b_m^{t-i} \Delta_{i-1} \\ &= (-1)^t \sum_{i=2}^t (-1)^{t-i} b_m^{t-i-(t-1)} \Delta_{i-1} x^{t-i} \\ &= \sum_{i=2}^t (-1)^t (-1)^{t-i} b_m^{1-i} \Delta_{i-1} x^{t-i} \\ &= \sum_{i=2}^t (-1)^i b_m^{1-i} \Delta_{i-1} x^{t-i}. \end{aligned}$$

Therefore $q(x) = \sum_{i=2}^t (-1)^i b_m^{1-i} \Delta_{i-1} x^{t-i} = d_0 + d_1 x + d_2 x^2 + \dots + d_{n-m} x^{n-m}$, so that,

$$\begin{aligned} d_0 &= (-1)^t b_m^{1-t} \Delta_{t-1} \\ &= (-1)^{(n-m+2)} b_m^{1-(n-m+2)} \Delta_{(n-m+2)-1} \\ &= (-1)^{(n-m+2)} b_m^{m-n-1} \Delta_{n-m+1}, \\ d_1 &= (-1)^{t-1} b_m^{1-(t-1)} \Delta_{(t-1)-1} \\ &= (-1)^{t-1} b_m^{2-t} \Delta_{t-2}, \\ d_2 &= (-1)^{t-2} b_m^{1-(t-2)} \Delta_{(t-2)-1} \\ &= (-1)^{t-2} b_m^{3-t} \Delta_{t-3}, \\ &\vdots \\ d_{n-m-1} &= (-1)^{t-(n-m-1)} b_m^{1-(t-(n-m-1))} \Delta_{(t-(n-m-1))-1} \\ &= (-1)^{(n-m+2)-(n-m-1)} b_m^{1-((n-m+2)-(n-m-1))} \Delta_{((n-m+2)-(n-m-1))-1} \\ &= (-1)^3 b_m^{-2} \Delta_2 \\ d_{n-m} &= (-1)^{t-(n-m)} b_m^{1-(t-(n-m))} \Delta_{(t-(n-m))-1} \\ &= (-1)^{(n-m+2)-(n-m)} b_m^{1-((n-m+2)-(n-m))} \Delta_{((n-m+2)-(n-m))-1} \\ &= (-1)^2 b_m^{-1} \Delta_1. \end{aligned}$$

That is

$$(4.14) \quad d_j = (-1)^{(t-j)} b_m^{(j+1)-t} \Delta_{t-(j+1)}, \quad j = 0, 1, \dots, n-m.$$

Since $q(x) = \sum_{j=0}^{n-m} d_j x^j$ and $t = n - m + 2$ we arrive at:

$$\begin{aligned} q(x) &= \sum_{j=0}^{n-m} ((-1)^{t+j} b_m^{j+1-t} \Delta_{t-(j+1)}) x^j \\ &= \sum_{j=0}^{n-m} ((-1)^{(n-m+2)+j} b_m^{j+1-(n-m+2)} \Delta_{(n-m+2)-(j+1)}) x^j. \end{aligned}$$

Therefore: $q(x) = \sum_{j=0}^{n-m} ((-1)^{n-m+j} b_m^{j-(n-m+1)} \Delta_{(n-m+1)-j}) x^j$. \square

5. DETERMINANT OF THE HESSENBERG-TOEPLITZ MATRIX

Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0$ and $g(x) = b_m x^m - b_{m-1} x^{m-1} - \dots - b_2 x^2 - b_1 x - b_0$ be polynomials in $F[x]$ degree n and m respectively. In special case, if $b_{m-1} \neq 0$ in F , and

$$(5.1) \quad \begin{aligned} a_n &= -b_{m-1}, \quad a_{n-1} = -b_{m-2}, \quad \dots, \quad a_{n-m+2} = -b_1, \quad a_{n-m+1} = -b_0, \\ a_{n-m} &= 0, \quad a_{n-m-1} = 0, \quad \dots, \quad a_1 = 0, \quad a_0 = 0. \end{aligned}$$

then

$$\begin{aligned} f(x) &= 0 + 0x + \dots + 0x^{n-m} - b_{m-m} x^{n-m+1} - \dots - b_{m-2} x^{n-1} - b_{m-1} x^n \\ &= -b_{m-m} x^{n-m+1} - \dots - b_{m-2} x^{n-1} - b_{m-1} x^n. \end{aligned}$$

Consider the system of linear equation (2.2) is change to

$$(5.2) \quad \begin{aligned} -b_{2m-n} d_{n-m} - b_{2m-n+1} d_{n-m-1} - \dots - b_{m-1} d_1 + b_m d_0 &= -b_{2m-n-1}, \\ -b_{2m-n+1} d_{n-m} - b_{2m-n+2} d_{n-m-1} - \dots + b_m d_1 &= -b_{2m-n}, \\ &\vdots \\ -b_{m-1} d_{n-m} + b_m d_{n-m-1} &= -b_{m-2}, \\ b_m d_{n-m} &= -b_{m-1}, \end{aligned}$$

we define $b_j = 0$, for $j < 0$. Therefore the Hessenberg matrix of (4.8) is change to:

$$(5.3) \quad \mathbf{H}_t = \begin{pmatrix} -b_{m-1} & b_m & 0 & \dots & \dots & 0 \\ -b_{m-2} & -b_{m-1} & b_m & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ -b_{2m-n} & -b_{2m-n+1} & & \ddots & b_m & 0 \\ -b_{2m-n-1} & -b_{2m-n} & -b_{2m-n+1} & \dots & -b_{m-1} & b_m \\ 0 & x^{n-m} & x^{n-m-1} & \dots & x & 1 \end{pmatrix}.$$

As in (4.9), the determinant of this matrix is:

$$\begin{aligned} \det(\mathbf{H}_t) &= 1\Delta_{t-1} - xb_m \Delta_{t-2} + x^2 b_m^2 \Delta_{t-3} + \dots - x^{n-m} b_m^{n-m} \Delta_1 + 0 \\ &= \sum_{i=2}^t (-1)^{t-i} x^{t-i} b_m^{t-i} \Delta_{i-1}. \end{aligned}$$

where

$$\Delta_1 = |-b_{m-1}|, \Delta_2 = \begin{vmatrix} -b_{m-1} & b_m \\ -b_{m-2} & -b_{m-1} \end{vmatrix}, \Delta_3 = \begin{vmatrix} -b_{m-1} & b_m & 0 \\ -b_{m-2} & -b_{m-1} & b_m \\ -b_{m-3} & -b_{m-2} & -b_{m-1} \end{vmatrix}, \dots$$

$$(5.4) \quad \Delta_{t-1} = \begin{vmatrix} -b_{m-1} & b_m & 0 & \dots & 0 \\ -b_{m-2} & -b_{m-1} & b_m & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ -b_{2m-n} & -b_{2m-n+1} & & \ddots & b_m \\ -b_{2m-n-1} & -b_{2m-n} & -b_{2m-n+1} & \dots & -b_{m-1} \end{vmatrix}.$$

The following is the main result:

Theorem 5.1. *If $\Delta_k = \begin{vmatrix} -b_{m-1} & b_m & 0 & \dots & 0 \\ -b_{m-2} & -b_{m-1} & b_m & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ -b_{m-k+1} & -b_{m-k+2} & & \ddots & b_m \\ -b_{m-k} & -b_{m-k+1} & -b_{m-k+2} & \dots & -b_{m-1} \end{vmatrix}$ then*

$$\Delta_k = (-1)^k b_m^k \sum_{i=1}^k t_i b_{m-(k+1)+i} = (-1)^k b_m^k \sum_{j=1}^{k-1} t_{k-j} b_{(m-1)-j}$$

where $\{t_r\}$ is the linear recurrent sequence, $t_1 = \frac{1}{b_m}$ and $t_r = \frac{1}{b_m} \sum_{i=1}^{r-1} b_{m-i} t_{r-i}$ for $r = 2, 3, \dots$

Proof. From (5.4)

$$\Delta_{t-1} = \begin{vmatrix} -b_{m-1} & b_m & 0 & \dots & 0 \\ -b_{m-2} & -b_{m-1} & b_m & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ -b_{2m-n} & -b_{2m-n+1} & & \ddots & b_m \\ -b_{2m-n-1} & -b_{2m-n} & -b_{2m-n+1} & \dots & -b_{m-1} \end{vmatrix},$$

where $t = n - m + 2$. Let $k = n - m + 1$. Thus this determinant change to:

$$\Delta_k = \begin{vmatrix} -b_{m-1} & b_m & 0 & \dots & 0 \\ -b_{m-2} & -b_{m-1} & b_m & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ -b_{m-k+1} & -b_{m-k+2} & & \ddots & b_m \\ -b_{m-k} & -b_{m-k+1} & -b_{m-k+2} & \dots & -b_{m-1} \end{vmatrix}.$$

According to Theorem 4.6,

$$(5.5) \quad q(x) = \sum_{i=0}^{n-m} \left((-1)^{n-m-i} b_m^{i-(n-m+1)} \Delta_{(n-m+1)-i} \right) x^i.$$

and Corollary 2.2 asserted that

$$q(x) = \sum_{i=0}^{n-m} \left(\sum_{j=0}^{n-m-i} t_{n-m-i+1-j} a_{n-j} \right) x^i.$$

where $\{t_r\}$ is the linear recurrent sequence, $t_1 = \frac{1}{b_m}$, and $t_r = \frac{1}{b_m} \sum_{i=1}^{r-1} b_{m-i} t_{r-i}$ for $r = 2, 3, \dots$. In this case, by (5.1), we get

$$(5.6) \quad q(x) = - \sum_{i=0}^{n-m} \left(\sum_{j=0}^{n-m-i} t_{n-m-i+1-j} b_{(m-1)-j} \right) x^i.$$

By the uniqueness of the quotient $q(x) = d_0 + d_1 x + \dots + d_{n-m} x^{n-m}$, from (5.5) and (5.6) we have

$$\begin{aligned} q(x) &= \sum_{i=0}^{n-m} \left((-1)^{n-m-i} b_m^{i-(n-m+1)} \Delta_{(n-m+1)-i} \right) x^i \\ &= - \sum_{i=0}^{n-m} \left(\sum_{j=0}^{n-m-i} t_{n-m-i+1-j} b_{(m-1)-j} \right) x^i. \end{aligned}$$

Equating the coefficients of the quotient polynomial, we get

$$\begin{aligned} d_0 &= (-1)^{n-m-0} b_m^{0-(n-m+1)} \Delta_{(n-m+1)-0} \\ (5.7) \quad &= - \sum_{j=0}^{n-m-0} t_{n-m-0+1-j} b_{(m-1)-j} \\ d_1 &= (-1)^{n-m-1} b_m^{1-(n-m+1)} \Delta_{(n-m+1)-1} \\ &= - \sum_{j=0}^{n-m-1} t_{n-m-1+1-j} b_{(m-1)-j} \\ &\vdots \\ d_{n-m-1} &= (-1)^{n-m-(n-m-1)} b_m^{(n-m-1)-(n-m+1)} \Delta_{(n-m+1)-(n-m-1)} \\ &= - \sum_{j=0}^{n-m-(n-m-1)} t_{n-m-(n-m-1)+1-j} b_{(m-1)-j} \\ d_{n-m} &= (-1)^{n-m-(n-m)} b_m^{(n-m)-(n-m+1)} \Delta_{(n-m+1)-(n-m)} \\ &= - \sum_{j=0}^{n-m-(n-m-1)} t_{n-m-(n-m)+1-j} b_{(m-1)-j}. \end{aligned}$$

We claim that

$$(5.8) \quad \Delta_k = (-1)^k b_m^k \sum_{i=1}^k t_i b_{m-(k+1)+i}.$$

Equation (5.8) can proof by mathematical induction on k .

For $k = 1$, consider

$$d_{n-m} = (-1)^0 b_m^{-1} \Delta_1 = - \sum_{j=0}^0 t_{1-1} b_{(m-1)-j}$$

Since $(-1)^0 b_m^{-1} \Delta_1 = \frac{1}{b_m} | - b_{m-1} | = -t_1 b_{m-1}$ implies that

$$\Delta_1 = -b_m t_1 b_{m-1} = (-1)^1 b_m^1 \sum_{i=1}^1 t_i b_{m-(1+1)-i}$$

Thus the equation (5.8) is true for $k = 1$.

Assume that the equation (5.8) is true for all order of the determinant in (5.4) $k < n - m + 1$, we must show that it true for $k = n - m + 1$. From (5.7)

$$d_0 = (-1)^{n-m} b_m^{-(n-m+1)} \Delta_{(n-m+1)} = - \sum_{j=0}^{n-m} t_{n-m+1-j} b_{(m-1)-j}$$

implies

$$\begin{aligned} \Delta_{(n-m+1)} &= (-1)^{1-n+m} b_m^{n-m+1} \sum_{j=0}^{n-m} t_{n-m+1-j} b_{(m-1)-j} \\ &= (-1)^{n-m+1} b_m^{n-m+1} \sum_{j=0}^{n-m} t_{n-m+1-j} b_{(m-1)-j} \end{aligned}$$

If $k = n - m + 1$ then we have:

$$\Delta_k = (-1)^k b_m^k \sum_{j=0}^{n-m} t_{k-j} b_{(m-1)-j}$$

Let $i = k - j$, we see that

$$\begin{aligned} \Delta_k &= (-1)^k b_m^k \sum_{j=0}^{n-m} t_{k-j} b_{(m-1)-j} = (-1)^k b_m^k \sum_{i=k}^1 t_i b_{(m-1)-(k-i)} \\ &= (-1)^k b_m^k \sum_{i=1}^k t_i b_{m-(1+k)+i} \end{aligned}$$

as claim, and the theorem was proved. \square

Corollary 5.2. If $\Delta_k = \begin{vmatrix} b_{m-1} & -b_m & 0 & \dots & 0 \\ b_{m-2} & b_{m-1} & -b_m & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ b_{m-k+1} & b_{m-k+2} & & \ddots & -b_m \\ b_{m-k} & b_{m-k+1} & b_{m-k+2} & \dots & b_{m-1} \end{vmatrix}$ then

$$(5.9) \quad \Delta_k = b_m^k \sum_{i=1}^k t_i b_{m-(k+1)+i} = b_m^k \sum_{j=1}^{k-1} t_{k-j} b_{(m-1)-j}$$

where $\{t_r\}$ is the linear recurrent sequence, $t_1 = \frac{1}{b_m}$ and $t_r = \frac{1}{b_m} \sum_{i=1}^{r-1} b_{m-i} t_{r-i}$ for $r = 2, 3, \dots$

Proof. From Theorem 5.1 assert that

$$\begin{aligned} \Delta_k &= \begin{vmatrix} -b_{m-1} & b_m & 0 & \dots & 0 \\ -b_{m-2} & -b_{m-1} & b_m & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ -b_{m-k+1} & -b_{m-k+2} & & \ddots & b_m \\ -b_{m-k} & -b_{m-k+1} & -b_{m-k+2} & \dots & -b_{m-1} \end{vmatrix} \\ (5.10) \quad &= (-1)^k b_m^k \sum_{i=1}^k t_i b_{m-(k+1)+i} = (-1)^k b_m^k \sum_{j=1}^{k-1} t_{k-j} b_{(m-1)-j} \end{aligned}$$

where $\{t_r\}$ is the linear recurrent sequence, $t_1 = \frac{1}{b_m}$ and $t_r = \frac{1}{b_m} \sum_{i=1}^{r-1} b_{m-i} t_{r-i}$ for $r = 2, 3, \dots$. Multiplying (5.10) both sides by $(-1)^k$, we see that

$$\begin{aligned}
& (-1)^k \begin{vmatrix} -b_{m-1} & b_m & 0 & \dots & 0 \\ -b_{m-2} & -b_{m-1} & b_m & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ -b_{m-k+1} & -b_{m-k+2} & & \ddots & b_m \\ -b_{m-k} & -b_{m-k+1} & -b_{m-k+2} & \dots & -b_{m-1} \end{vmatrix} \\
&= \begin{vmatrix} b_{m-1} & -b_m & 0 & \dots & 0 \\ b_{m-2} & b_{m-1} & -b_m & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ b_{m-k+1} & b_{m-k+2} & & \ddots & -b_m \\ b_{m-k} & b_{m-k+1} & b_{m-k+2} & \dots & b_{m-1} \end{vmatrix} \\
&= (-1)^{2k} b_m^k \sum_{i=1}^k t_i b_{m-(k+1)+i} = (-1)^{2k} b_m^k \sum_{j=1}^{k-1} t_{k-j} b_{(m-1)-j} \\
&= b_m^k \sum_{i=1}^k t_i b_{m-(k+1)+i} = b_m^k \sum_{j=1}^{k-1} t_{k-j} b_{(m-1)-j}.
\end{aligned}$$

The corollary was proved. \square

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W. Wanicharpichat
 Department of Mathematics
 Faculty of Science
 Naresuan University
 Phitsanulok 65000, Thailand.
 E-mail: wiwatw@nu.ac.th